Remarks on Smarr's mass formula
in the presence of both electric and magnetic charges

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The present paper clarifies how to use correctly Tomimatsu's representation of Komar integrals for consistently taking into account the contribution of magnetic charges in the generalized Smarr mass formula. In all three examples of the dyonic solutions considered by us, the sum of the two electromagnetic terms in that formula can be cast into the form $\bar{\mathcal{Q}}\Phi^H_{ext}$, where $\mathcal{Q}$ is the complex charge and Φ^H_{ext} the complex extension of the electric potential.

I. INTRODUCTION

The well-known Smarr formula [1] relates concisely various physical characteristics of a single black hole, and in its original version the black hole can only carry the electric charge, being free of the magnetic one. Smarr's mass formula and some of its possible generalizations were extensively analyzed and discussed by Carter [2] with the aid of the Komar integrals [3], and later Tomimatsu [4] succeeded in employing the Ernst formalism of complex potentials [5] for presenting Carter's general expressions in a simple and elegant form suitable for the use even in the multi-black-hole spacetimes. The specific problem studied by Tomimatsu in [4] was application of the Ernst-Harrison charging transformation [5, 6] to the celebrated double-Kerr solution of Kramer and Neugebauer [7], and in particular he established that in that kind of binary systems the usual Smarr formula was not satisfied generically by a charged spinning black-hole constituent, what he associated with the emergence of a Dirac string signaling about the presence of magnetic charges. It should be emphasized that the correctness of Tomimatsu's integral formulas was checked both analytically and numerically in numerous papers devoted to binary black-hole systems; nonetheless, it appears that the recent articles [8, 9] dealing with a reparametrization of the Ernst-Manko-Ruiz (EMR) solution for charged counterrotating sources [10, 11] still cast doubt on the validity of these formulas since the authors of [8, 9] presented some dubious "enhanced" mass relations which they attributed to Tomimatsu and which they used for obtaining physically inconsistent results that have been commented in [12]. In this respect, it seems desirable, on the one hand, to figure out the precise source of errors lying behind the faulty results obtained in [8, 9] and, on the other hand, to show how the correct use of Tomimatsu's formulas is able to describe properly the electric and magnetic charge contributions in the generalized Smarr formula.

In order to make our discussion of the stationary dyonic dihole case – the principle one treated erroneously by the authors of [9] – more comprehensible to the reader, in the next section we will first consider the application of Tomimatsu's formulas to the dyonic Kerr-Newman (KN) solution [2]; it will be seen that this case does not represent any problems with regard to the generalized Smarr formula and, moreover, can be helpful for identifying the basic error of the paper [9] violating its physical consistency. In Sec. III the 5-parameter solution for a stationary dyonic dihole will be reexamined with the aid of Tomimatsu's formulas within the lines of the paper [13], and in particular it will be explained why the Smarr formula in the dyonic case may only slightly differ from the zero magnetic charge case. A dyonic generalization of the Bretón-Manko (BM) solution [14, 15] will be discussed in Sec. IV as a non-trivial extension of Sec. III, with the idea to demonstrate possible complications that may arise after the introduction of magnetic charge into the solution. Concluding remarks are given in Sec. V.

II. THE DYONIC KN SOLUTION

In this section we will show that in the case of the dyonic KN solution [2], which is a one-parameter generalization of the usual KN spacetime [16], the Tomimatsu's integrals are able to lead us straightforwardly to the desired form of the generalized Smarr formula. Since these integrals were originally designed for the work in the Weyl-Papapetrou cylindrical coordinates (ρ, z), it would be likely to have a representation of the dyonic KN solution in the latter coordinates. Note that this solution can be derived in a rigorous way by means of Sibgatullin's integral method [17, 18], starting from the axis data

$$\mathcal{E}(\rho=0, z) = \frac{z - M - ia}{z + M - ia}, \quad \Phi(\rho=0, z) = \frac{Q + i\mathcal{B}}{z + M - ia}, \tag{1}$$

where M, a, Q and $\mathcal{B}$ are real parameters standing, respectively, for the mass, angular momentum per unit mass, electric and magnetic charges of the source. Then the resulting Ernst potentials $\mathcal{E}$ and Φ, as well as the metric functions f, γ and ω in the stationary axisymmetric line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{2}$$

together with the electric A_t and magnetic A_φ components of the electromagnetic 4–potential have the form [19, 20]

$$\begin{aligned}
\mathcal{E} &= \frac{\sigma x - M - iay}{\sigma x + M - iay}, \quad \Phi = \frac{Q + i\mathcal{B}}{\sigma x + M - iay}, \\
f &= \frac{\sigma^2(x^2 - 1) - a^2(1 - y^2)}{(\sigma x + M)^2 + a^2 y^2}, \quad e^{2\gamma} = \frac{\sigma^2(x^2 - 1) - a^2(1 - y^2)}{\sigma^2(x^2 - y^2)}, \\
\omega &= -\frac{a(1 - y^2)[2M(\sigma x + M) - Q^2 - \mathcal{B}^2]}{\sigma^2(x^2 - 1) - a^2(1 - y^2)}, \\
A_t &= -\frac{Q(\sigma x + M) - a\mathcal{B}y}{(\sigma x + M)^2 + a^2 y^2}, \\
A_\varphi &= b_0 - \mathcal{B}y + \frac{a(1 - y^2)[Q(\sigma x + M) - a\mathcal{B}y)]}{(\sigma x + M)^2 + a^2 y^2}, \\
\sigma &= \sqrt{M^2 - a^2 - Q^2 - \mathcal{B}^2},
\end{aligned} \tag{3}$$

where (x, y) are related to (ρ, z) by

$$x = \frac{1}{2\sigma}(r_+ + r_-), \quad y = \frac{1}{2\sigma}(r_+ - r_-), \quad r_\pm = \sqrt{\rho^2 + (z \pm \sigma)^2}, \tag{4}$$

or, inversely,

$$\rho = \sigma\sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \sigma xy. \tag{5}$$

It should be noted that the expression for A_φ in (3) contains the integration constant b_0 whose particular choice, as will be seen later on, is very important for obtaining a correct generalization of Smarr's mass formula.

The above formulas describe a non-extreme black hole when $M^2 > a^2 + Q^2 + \mathcal{B}^2$, and Tomimatsu's integrals for M, $J = Ma$ and Q, supplemented with the formula for the magnetic charge $\mathcal{B}$, have the form [4]

$$M = -\frac{1}{4}\int_H \omega\Omega_{,z}dz, \tag{6}$$

$$J = \frac{1}{4}\int_H \omega\left[-1 - \tfrac{1}{2}\omega\Omega_{,z} + \tilde{A}_\varphi A'_{\varphi,z} + (A_\varphi A'_\varphi)_{,z}\right]dz, \tag{7}$$

$$Q = \frac{1}{2}\int_H \omega A'_{\varphi,z}dz, \tag{8}$$

$$\mathcal{B} = \frac{1}{2}\int_H \omega A_{t,z}dz, \tag{9}$$

with $\Omega = \mathrm{Im}(\mathcal{E})$, $A'_\varphi = \mathrm{Im}(\Phi)$, $\tilde{A}_\varphi = A_\varphi + \omega A_t$, all the functions entering (6)-(9) to be evaluated on the horizon which in the (ρ, z) coordinates is defined as the subset $\rho = 0$, $-\sigma < z < \sigma$ of the z-axis (see Fig. 1). There are also four more black-hole characteristics involved in Tomimatsu's paper, which are [2, 4]

$$\kappa = \sqrt{-\omega^{-2}e^{-2\gamma}}, \quad S = 4\pi\sigma\sqrt{-\omega^2 e^{2\gamma}}, \quad \Omega^H = \omega^{-1}, \quad \Phi^H = -A_t - \Omega^H A_\varphi, \tag{10}$$

κ being the surface gravity, S the horizon's area, Ω^H its angular velocity and Φ^H the electric potential, and all of them are constant quantities because the functions ω, γ and $\tilde{A}_\varphi$ take constant values on the horizon.

We now turn to the mass formula of Smarr which (in the absence of magnetic charge $\mathcal{B}$) reads

$$M = \frac{1}{4\pi}\kappa S + 2J\Omega^H + Q\Phi^H = \sigma + 2J\Omega^H + Q\Phi^H. \tag{11}$$

Solving (11) for J and recalling that $\omega = 1/\Omega^H$, we get

$$J = \frac{\omega}{2}(-\sigma + M - Q\Phi^H), \tag{12}$$

whence it follows that it is equation (7) for J which actually contains Smarr's formula: the correspondence of the first two terms on the right hand sides of (12) and (7) is trivially seen, while the correspondence between the third terms is readily established if one takes into account that $\tilde{A}_\varphi = -\omega \Phi^H$. For the usual KN solution, the contribution of the fourth term on the right of (7) is zero if the constant b_0 in the potential A_φ is set equal to zero, which means that even in the absence of magnetic charge the correct choice of b_0 is needed for the consistency of Tomimatsu's formula (7).

As was observed for instance in [21], Smarr's mass formula in the presence of magnetic charge $\mathcal{B}$ is expected to have the structure

$$M = \frac{1}{4\pi}\kappa S + 2J\Omega^H + Q\Phi^H + \mathcal{B}\Phi_m^H, \tag{13}$$

for taking into account the magnetic charge contribution via some magnetic potential Φ_m^H. Therefore it will be instructive to work out the dyonic KN case in full detail using formulas (3)-(9). For this purpose we may start with the evaluation of the potentials $\mathcal{E}$, Φ, A_t, A_φ (with $b_0 = 0$) and the metric functions (3) on the horizon where $r_\pm = \sigma \pm z$ and hence $x = 1$, $y = z/\sigma$. The resulting expressions for $\mathcal{E}$, Φ, A_t and A_φ on the horizon are

$$\mathcal{E} = \frac{\sigma^2 - M\sigma - iaz}{\sigma^2 + M\sigma - iaz}, \quad \Phi = \frac{\sigma(Q + i\mathcal{B})}{\sigma^2 + M\sigma - iaz},$$
$$A_t = -\frac{Q\sigma^2(M + \sigma) - a\mathcal{B}\sigma z}{\sigma^2(M + \sigma)^2 + a^2 z^2}, \quad A_\varphi = -\frac{\mathcal{B}z}{\sigma} + \frac{a(\sigma^2 - z^2)[Q\sigma(M + \sigma) - a\mathcal{B}z]}{\sigma[\sigma^2(M + \sigma)^2 + a^2 z^2]} \tag{14}$$

(the reader is reminded that the constant b_0 in the potential A_φ is set equal to zero), while both γ and ω on the horizon take constant values

$$e^{2\gamma} = -\frac{a^2}{\sigma^2}, \quad \omega = \frac{(M + \sigma)^2 + a^2}{a}. \tag{15}$$

Then from (14) we get the form of Ω and A'_φ,

$$\Omega = \text{Im}(\mathcal{E}) = -\frac{2Ma\sigma z}{\sigma^2(M + \sigma)^2 + a^2 z^2}, \quad A'_\varphi = \text{Im}(\Phi) = \frac{\mathcal{B}\sigma^2(M + \sigma) + aQ\sigma z}{\sigma^2(M + \sigma)^2 + a^2 z^2}, \tag{16}$$

and (14) and (15) permit us to see that the combination $A_\varphi + \omega A_t$ is constant:

$$\tilde{A}_\varphi = -\frac{Q(M + \sigma)}{a}. \tag{17}$$

From (15) and (10) we find the expressions for the surface gravity, horizon's area and horizon's angular velocity:

$$\kappa = \frac{\sigma}{(M + \sigma)^2 + a^2}, \quad S = 4\pi[(M + \sigma)^2 + a^2], \quad \Omega^H = \frac{a}{(M + \sigma)^2 + a^2}, \tag{18}$$

while the form of the electric potential Φ^H is obtainable from (10), (14) and (18), yielding

$$\Phi^H = \frac{Q(M + \sigma)}{(M + \sigma)^2 + a^2}. \tag{19}$$

Since ω and $\tilde{A}_\varphi$ are constant quantities in Tomimatsu's formulas (6)-(9), the integration there just reduces to evaluating the differences of the functions at the points $z = +\sigma$ and $z = -\sigma$. Thus, for example, it is easy to check that the relation (6) for M is satisfied identically,

$$M = -\frac{1}{4}\omega[\Omega(z = +\sigma) - \Omega(z = -\sigma)] = M, \tag{20}$$

and the same check can be readily performed in the formulas (8) and (9) too. Obviously, the magnetic potential Φ_m^H should be found from the Tomimatsu's formula (7) for J, and it is defined by the fourth term in the integrand. Then, taking into account that

$$(A_\varphi A'_\varphi)_{z=+\sigma} - (A_\varphi A'_\varphi)_{z=-\sigma} = -\frac{2\mathcal{B}^2(M + \sigma)}{(M + \sigma)^2 + a^2}, \tag{21}$$

the formula for J can be written as

$$J = \frac{\omega}{2}(-\sigma + M - Q\Phi^H - \mathcal{B}\Phi_m^H), \tag{22}$$

after the introduction of the magnetic potential Φ_m^H of the form

$$\Phi_m^H = \frac{\mathcal{B}(M + \sigma)}{(M + \sigma)^2 + a^2}. \tag{23}$$

Recalling that $\sigma = \kappa S/4\pi$ and $\omega = 1/\Omega^H$, we eventually arrive at the conclusion that the choice $b_0 = 0$ in the potential A_φ from (3) ensures the verification of the generalized Smarr formula by the KN dyon, Tomimatsu's formula (7) describing correctly the angular momentum $J = Ma$ of the solution.

It is worth remarking that if the potential A_φ had a non-zero b_0, then the third and fourth terms on the right hand side of (7) would have modified equation (22) in the following way:

$$J = \frac{\omega}{2}(-\sigma + M - Q\Phi^H - \mathcal{B}\Phi_m^H) + Qb_0, \tag{24}$$

thus violating the generalized Smarr formula (13). Precisely for that reason the constant b_0 in (3) must be set equal to zero; however, in some more complex dyonic solutions the constant b_0, as will be seen in the next two sections, must be assigned non-zero values to ensure consistent verification of the mass formula (13).

It should be also noted that the values of the potentials Φ^H and Φ_m^H of the KN dyon suggest that the last two (electromagnetic) terms in the generalized Smarr formula (13) can be combined in one expression. Indeed, from (19) and (23) we get

$$Q\Phi^H + \mathcal{B}\Phi_m^H = \frac{(Q^2 + \mathcal{B}^2)(M + \sigma)}{(M + \sigma)^2 + a^2}, \tag{25}$$

so that by introducing the complex charge $\mathcal{Q}$ and the extended electric potential Φ_{ext}^H via the formulas

$$\mathcal{Q} = Q + i\mathcal{B}, \quad \Phi_{ext}^H = \frac{\mathcal{Q}(M + \sigma)}{(M + \sigma)^2 + a^2}, \tag{26}$$

the left hand side of (25) takes the form $\bar{\mathcal{Q}}\Phi_{ext}^H$, and the generalized Smarr formula (13) rewrites as

$$M = \frac{1}{4\pi}\kappa S + 2J\Omega^H + \bar{\mathcal{Q}}\Phi_{ext}^H, \tag{27}$$

thus being only slightly different from the usual mass formula (11). In what follows we shall see that the above form of the generalized mass relation may also hold in the binary systems of interacting dyons.

III. THE EMR DYONIC DIHOLE

A more complicated dyonic model would be a pair of counterrotating identical KN dyons whose electric (as well as magnetic) charges are equal in absolute values but have opposite signs. Such model is described by the EMR metric [10] which contains as a particular case the Emparan-Teo static electric dihole solution [22]), so that the more general model to be considered below represents a stationary dyonic dihole with equatorial antisymmetry [23]. A physical parametrization of the EMR solution involving Komar quantities has been worked out in the paper [13], the solution's potentials $\mathcal{E}$ and Φ in that parametrization having the form

$$\begin{aligned}
\mathcal{E} &= \frac{A - B}{A + B}, \quad \Phi = \frac{C}{A + B}, \\
A &= R^2(M^2 - |\mathcal{Q}|^2\nu)(R_+ - R_-)(r_+ - r_-) + 4\sigma^2(M^2 + |\mathcal{Q}|^2\nu)(R_+ - r_+)(R_- - r_-) \\
&\quad + 2R\sigma[R\sigma(R_+r_- + R_-r_+) + iMa\mu(R_+r_- - R_-r_+)], \\
B &= 2MR\sigma[R\sigma(R_+ + R_- + r_+ + r_-) - (2M^2 - iMa\mu)(R_+ - R_- - r_+ + r_-)], \\
C &= 2C_0R\sigma[(R + 2\sigma)(R\sigma - 2M^2 - iMa\mu)(r_+ - R_-) + (R - 2\sigma) \\
&\quad \times (R\sigma + 2M^2 + iMa\mu)(r_- - R_+)], \\
R_\pm &= \sqrt{\rho^2 + (z + \tfrac{1}{2}R \pm \sigma)^2}, \quad r_\pm = \sqrt{\rho^2 + (z - \tfrac{1}{2}R \pm \sigma)^2},
\end{aligned} \tag{28}$$

where

$$\mathcal{Q} = Q + i\mathcal{B}, \quad |\mathcal{Q}|^2 = Q^2 + \mathcal{B}^2,$$

$$\sigma = \sqrt{M^2 - \left(\frac{M^2 a^2[(R+2M)^2 + 4|\mathcal{Q}|^2]}{[M(R+2M)+|\mathcal{Q}|^2]^2} + |\mathcal{Q}|^2\right)\frac{R-2M}{R+2M}}, \tag{29}$$

and μ, ν and C_0 are dimensionless constant quantities defined as

$$\mu = \frac{R^2 - 4M^2}{M(R+2M)+|\mathcal{Q}|^2}, \quad \nu = \frac{R^2 - 4M^2}{(R+2M)^2 + 4|\mathcal{Q}|^2},$$

$$C_0 = -\frac{\mathcal{Q}(R^2 - 4M^2 + 2iMa\mu)}{(R+2M)(R^2 - 4\sigma^2)}. \tag{30}$$

The real constants M, Q and $\mathcal{B}$ are the mass, electric and magnetic charges of the upper KN dyon, whose angular momentum is Ma; the respective characteristics of the lower dyon are M, $-Q$, $-\mathcal{B}$ and $-Ma$; R is the separation coordinate distance (see Fig. 2(a)). Remarkably, like in the case of a single KN dyon, the charges Q and $\mathcal{B}$ enter formulas (28)-(30) only in the combinations $\mathcal{Q} = Q + i\mathcal{B}$ and $\mathcal{Q}\bar{\mathcal{Q}} \equiv |\mathcal{Q}|^2 = Q^2 + \mathcal{B}^2$.

The corresponding metric coefficients f, γ and ω are given by the expressions

$$f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16R^4\sigma^4 R_+ R_- r_+ r_-}, \quad \omega = -\frac{\text{Im}[2G(\bar{A}+\bar{B})+C\bar{I}]}{A\bar{A}-B\bar{B}+C\bar{C}},$$

$$G = -zB + R\sigma\{R(2M^2 - |\mathcal{Q}|^2\nu)(R_- r_- - R_+ r_+) + 2\sigma(2M^2 + |\mathcal{Q}|^2\nu)(r_+ r_- - R_+ R_-)$$

$$+M[(R+2\sigma)(R\sigma - 2M^2 + iMa\mu) + 2(R-2\sigma)|\mathcal{Q}|^2\nu](R_+ - r_-)$$

$$+M[(R-2\sigma)(R\sigma + 2M^2 - iMa\mu) - 2(R+2\sigma)|\mathcal{Q}|^2\nu](R_- - r_+)\},$$

$$I = -zC + 2C_0 M[R^2(2M^2 - 2\sigma^2 + iMa\mu)(R_+ r_+ + R_- r_-)$$

$$+2\sigma^2(R^2 - 4M^2 - 2iMa\mu)(R_+ R_- + r_+ r_-)] - C_0(R^2 - 4\sigma^2)$$

$$\times\{2M[R\sigma(R_+ r_- - R_- r_+) + (2M^2 + iMa\mu)(R_+ r_- + R_- r_+)] + R\sigma[R\sigma$$

$$\times(R_+ + R_- + r_+ + r_-) + (6M^2 + iMa\mu)(R_+ - R_- - r_+ + r_-) + 8MR\sigma]\}, \tag{31}$$

while the t and φ components of the electromagnetic 4-potential are given by the formulas

$$A_t = -\text{Re}\left(\frac{C}{A+B}\right), \quad A_\varphi = b_0 + \text{Im}\left(\frac{I}{A+B}\right), \tag{32}$$

where b_0 is a real constant whose particular value should be found from Tomimatsu's formula (7).

Similar to the previous case considered in Sec. II, it can be checked that Tomimatsu's formulas (6), (8) and (9) are satisfied identically by the solution (28)-(32) on both horizons, thus supporting the interpretation of the parameters M, Q and $\mathcal{B}$. At the same time, computations performed in the formula (7) for J on the upper horizon ($\rho = 0$, $\frac{1}{2}R - \sigma \leq z \leq \frac{1}{2}R + \sigma$) eventually lead to the equation

$$J = Ma + Q\mathcal{B} + b_0 Q, \tag{33}$$

whence we get

$$b_0 = -\mathcal{B}, \tag{34}$$

taking into account that $J = Ma$ by definition. On the other hand, at the lower horizon ($\rho = 0$, $-\frac{1}{2}R - \sigma \leq z \leq -\frac{1}{2}R + \sigma$) the equation for the determination of b_0 would take the form

$$-J = -Ma - Q\mathcal{B} - b_0 Q, \tag{35}$$

thus being also consistent with the choice (34) for b_0.

Note that it is precisely the absence of the constant b_0 in the potential A_φ of the paper [9] that led the authors of the latter to an unphysical redefinition $J - Q\mathcal{B}$ of the angular momentum throughout the solution, thus making their results physically inconsistent (see [12] for details).

The substitution $b_0 = -\mathcal{B}$ into (32) ensures the verification of the generalized Smarr formula (13). Indeed, the quantities κ, S, Ω^H, Φ^H for the upper dyon constituent can be shown to be determined by the expressions

$$\kappa = \frac{R\sigma[(R+2M)^2 + 4|\mathcal{Q}|^2]}{(R+2M)^2[2(M+\sigma)(MR+2M^2+|\mathcal{Q}|^2) - |\mathcal{Q}|^2(R-2M)]}, \tag{36}$$

$$S = \frac{4\pi}{R(R+2\sigma)}\left((R+2M)^2(M+\sigma)^2 + \frac{M^2a^2(R^2-4M^2)^2}{(MR+2M^2+|\mathcal{Q}|^2)^2}\right), \tag{37}$$

$$\Omega^H = \frac{Ma[2(M-\sigma)(MR+2M^2+|\mathcal{Q}|^2) - |\mathcal{Q}|^2(R-2M)]}{(4M^2a^2+|\mathcal{Q}|^4)(MR+2M^2+|\mathcal{Q}|^2)}, \tag{38}$$

$$\Phi^H = \frac{Q[|\mathcal{Q}|^2(M-\sigma)(MR+2M^2+|\mathcal{Q}|^2) + 2M^2a^2(R-2M)]}{(4M^2a^2+|\mathcal{Q}|^4)(MR+2M^2+|\mathcal{Q}|^2)}, \tag{39}$$

while for the magnetic potential Φ_m^H the fourth term of Tomimatsu's formula (7) gives

$$\Phi_m^H = \mathcal{B}\Phi^H/Q, \tag{40}$$

and it is straightforward to check that (36)-(40) satisfy (13) identically. Apparently, the lower dyon constituent satisfies (13) too, as the latter equation is invariant under the sign change $J \to -J$, $\Omega^H \to -\Omega^H$, $Q \to -Q$, $\Phi^H \to -\Phi^H$, $\mathcal{B} \to -\mathcal{B}$, $\Phi_m^H \to -\Phi_m^H$.

It is also clear that Φ^H and Φ_m^H can be combined in one potential of the form

$$\Phi_{ext}^H = \Phi^H + i\Phi_m^H = \frac{\mathcal{Q}[|\mathcal{Q}|^2(M-\sigma)(MR+2M^2+|\mathcal{Q}|^2) + 2M^2a^2(R-2M)]}{(4M^2a^2+|\mathcal{Q}|^4)(MR+2M^2+|\mathcal{Q}|^2)}, \tag{41}$$

in full analogy with the case of a sole KN dyon, and now the quantities (36)-(38) and (41) satisfy the generalized Smarr formula in the form (27) which can also be written as

$$M = \sigma + 2J\Omega^H + \bar{\mathcal{Q}}\Phi_{ext}^H. \tag{42}$$

Moreover, it is not difficult to see that Φ_{ext}^H is obtainable from the potential Φ^H of the $\mathcal{B} = 0$ specialization of the general 5-parameter solution by formally changing Q to $\mathcal{Q}$ and Q^2 to $|\mathcal{Q}|^2$, so that the generalized mass formula (27) could in principle be viewed as obtainable from the usual Smarr formula (11) via the above parameter substitution, supplemented with changing the resulting $\mathcal{Q}\Phi_{ext}^H$ to $\bar{\mathcal{Q}}\Phi_{ext}^H$.

Since the magnetic lines of force of a magnetic dipole formed by a pair of magnetic monopoles of opposite signs are somewhat distinctive from those of a magnetic dipole generated by a rotating electric charge, it would be interesting to illustrate how the magnetic charge in the 5-parameter EMR solution affects the magnetic lines of force of the 4-parameter stationary electric dihole ($\mathcal{B} = 0$). In Fig. 3 all plots are defined by the same values of M, a, Q and R; however, the values of $\mathcal{B}$ are different. The magnetically uncharged case is depicted in Fig. 3(a), and it is represented by two individual magnetic dipoles generated by the KN black-hole constituents. In Fig. 3(b) the dyonic constituents carry opposite magnetic charges that are 25 times less in magnitude than their electric charges, and the magnetic lines of force are seen perturbed by the additional magnetic field. By increasing twice the value of $\mathcal{B}$, the magnetic lines of force in Fig. 3(c) become already qualitatively those of two opposite magnetic charges.

IV. THE DYONIC BM SOLUTION

In this section we shall consider an interesting example of a binary dyonic system requiring a somewhat more subtle use of Tomimatsu's formulas than in the case of a dyonic dihole. Concretely, we are going to analyze a system composed of two identical counterrotating KN dyons carrying the same electric/magnetic charges (both in magnitude and signs). Such a system is described by the BM metric [14] in which the magnetic charge must be introduced by means of the same parameter change $Q \to \mathcal{Q}$, $Q^2 \to |\mathcal{Q}|^2$ as in the case of the EMR 5-parameter solution. The Ernst potentials $\mathcal{E}$ and Φ of the dyonic BM solution thus have the form

$$\mathcal{E} = \frac{A-B}{A+B}, \quad \Phi = \frac{C}{A+B},$$

$$A = (M^2-|\mathcal{Q}|^2)[4\sigma^2(R_+R_- + r_+r_-) + R^2(R_+r_+ + R_-r_-)] + [\sigma^2(R^2-4M^2+4|\mathcal{Q}|^2)$$
$$-M^2a^2R^2\mu](R_+r_- + R_-r_+) - 2iaMR\mu\sigma(MR+2M^2-|\mathcal{Q}|^2)(R_+r_- - R_-r_+),$$

$$B = 2MR\sigma\{R\sigma(R_+ + R_- + r_+ + r_-) - [2(M^2-|\mathcal{Q}|^2) + iMa\mu(MR+2M^2-|\mathcal{Q}|^2)]$$

$$\times (R_+ - R_- - r_+ + r_-)\},$$
$$C = \mathcal{Q}B/M,$$
$$R_\pm = \sqrt{\rho^2 + (z + \tfrac{1}{2}R \pm \sigma)^2}, \quad r_\pm = \sqrt{\rho^2 + (z - \tfrac{1}{2}R \pm \sigma)^2}, \tag{43}$$

where

$$\mathcal{Q} = Q + i\mathcal{B}, \quad |\mathcal{Q}|^2 = Q^2 + \mathcal{B}^2,$$
$$\sigma = \sqrt{M^2 - |\mathcal{Q}|^2 - M^2 a^2 \mu}, \quad \mu = \frac{R^2 - 4M^2 + 4|\mathcal{Q}|^2}{(MR + 2M^2 - |\mathcal{Q}|^2)^2}, \tag{44}$$

and the parameters M, $-a$, Q, $\mathcal{B}$ stand for the mass, angular momentum per unit mass, electric and magnetic charges of the upper dyon constituent, while all the characteristics of the lower dyon are the same, except that its angular momentum per unit mass has opposite sign, $+a$ (see Fig. 2(b)).

The metric functions f, γ and ω of the dyonic BM solution, together with the potentials A_t and A_φ describing the electromagnetic field, have the form

$$f = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A+B)(\bar{A}+\bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16R^4\sigma^4 R_+ R_- r_+ r_-}, \quad \omega = -\frac{\text{Im}[2G(\bar{A} + \bar{B}) + C\bar{I}]}{A\bar{A} - B\bar{B} + C\bar{C}},$$
$$G = -zB + R\sigma\{(2M^2 - |\mathcal{Q}|^2)[2\sigma(r_+ r_- - R_+ R_-) + R(R_- r_- - R_+ r_+)]$$
$$+ M(R + 2\sigma)[R\sigma - 2(M^2 - |\mathcal{Q}|^2) - iMa\mu(MR + 2M^2 - |\mathcal{Q}|^2)](R_+ - r_-)$$
$$+ M(R - 2\sigma)[R\sigma + 2(M^2 - |\mathcal{Q}|^2) + iMa\mu(MR + 2M^2 - |\mathcal{Q}|^2)](R_- - r_+)\},$$
$$I = \frac{\mathcal{Q}}{M}\{G + R|\mathcal{Q}|^2\sigma[2\sigma(r_+ r_- - R_+ R_-) + R(R_- r_- - R_+ r_+)]\},$$
$$A_t = -\text{Re}\left(\frac{C}{A+B}\right), \quad A_\varphi = b_0 + \text{Im}\left(\frac{I}{A+B}\right), \tag{45}$$

where we again introduced the arbitrary constant b_0 in the expression of A_φ.

On the upper and lower horizons ($\tfrac{1}{2}R - \sigma \le z \le \tfrac{1}{2}R + \sigma$ and $-\tfrac{1}{2}R - \sigma \le z \le -\tfrac{1}{2}R + \sigma$ parts of the z-axis, respectively), Tomimatsu's formulas (6), (8) and (9) are satisfied identically, thus confirming the interpretation of the parameters M, Q and $\mathcal{B}$ as the mass, electric charge and magnetic charge of each dyon constituent. As for the angular momentum of the upper dyon defined as $-J = -Ma$, Tomimatsu's formula (7) leads instead of an identity to the equation

$$-J = -Ma - Q\mathcal{B} + b_0 Q, \tag{46}$$

whence it follows immediately that the constant b_0 must have the form

$$b_0 = \mathcal{B} \tag{47}$$

in order to satisfy (7) identically on the upper horizon. Now, making use of the results of the paper [13], it is not difficult to show that the quantities κ, S, Ω^H and Φ^H of the upper horizon are given by the expressions

$$\kappa = \frac{R\sigma}{\Delta}, \quad S = \frac{4\pi\Delta}{R}, \quad \Omega^H = -\frac{Ma\mu(MR + 2M^2 - |\mathcal{Q}|^2)}{\Delta},$$
$$\Phi^H = \frac{Q[(R + 2M)(M + \sigma) - 2|\mathcal{Q}|^2]}{\Delta},$$
$$\Delta = 2M(R + 2M)(M + \sigma) - |\mathcal{Q}|^2(R + 4M + 2\sigma), \tag{48}$$

and to these we should aggregate the expression of the magnetic potential Φ_m^H corresponding to the fourth term in the formula (7):

$$\Phi_m^H = \frac{\mathcal{B}[(R + 2M)(M + \sigma) - 2|\mathcal{Q}|^2]}{\Delta}. \tag{49}$$

Then a simple check indicates that the quantities (48) and (49) verify the generalized Smarr formula (13) or, by introducing

$$\Phi_{ext}^H = \frac{\mathcal{Q}[(R + 2M)(M + \sigma) - 2|\mathcal{Q}|^2]}{\Delta}, \tag{50}$$

the formula (27).

At this point, turning to the lower horizon, it might seem plausible to make use of the symmetry of the dyonic configuration under consideration and just conclude that the lower dyon verifies the generalized mass formula (13) too as it has the same physical characteristics (48) and (49) as the upper one (albeit the sign change in Ω^H that does not affect (13) because J also changes its sign). However, in reality, coming to such an obvious physical conclusion is not straightforward at all, since Tomimatsu's formula (7) leads on the lower horizon to the equation

$$J = Ma + Q\mathcal{B} + b_0 Q, \tag{51}$$

which becomes an identity only at

$$b_0 = -\mathcal{B}, \tag{52}$$

this value of b_0 being different from the one in (47). This difference, which was absent in the previous case of the dyonic EMR solution, is explained by the fact that the term $b_0 Q$ on the right hand sides of (46) and (51) has the same sign since the BM dyons carry identical charges, while the charges of the EMR dyons are opposite in sign and hence the term $b_0 Q$ enters equations (33) and (35) with different signs, thus not causing problems of consistency.

To resolve the above (actually spurious) problem of two-valued b_0, we have to resort to the help of the well-known papers of Wu and Yang [24, 25] on the Dirac monopole [26] where for disposing of the Dirac string singularity two regions were used to define a pair of potentials A_φ, differing (under a gauge transformation) only by an additive constant (a more general system of various interacting magnetic monopoles, electrons and photons has been considered in [27] within the same basic concept). Wu and Yang's idea of exploiting the gauge freedom for adjusting appropriately the magnetic component A_φ of the electromagnetic 4-potential was implemented in the general relativistic theory by Semiz [28] who introduced for his specific purposes a double-valued constant corresponding to two different regions precisely in the context of the dyonic KN solution. Following the aforementioned papers, we shall define A_φ of the dyonic BM solution on two domains, the first one with its gauge determined by the value $b_0 = \mathcal{B}$ for $z \geq 0$, and the other one with the second gauge defined by the value $b_0 = -\mathcal{B}$ for $z < 0$. Such a redefinition of A_φ is not only likely but in fact required in order for the potential A_φ to be consistent with the symmetry of our dyonic configuration after the choice $b_0 = \mathcal{B}$ has been made on the upper horizon. Therefore, the constant b_0 in (45) must be finally assigned the double value

$$b_0 = \pm\mathcal{B}, \tag{53}$$

where the plus and minus signs correspond to the two domains used to define the potential A_φ. As a consequence, Tomimatsu's formula (7) works perfectly well on both horizons of the dyonic BM solution where the generalized mass relations (13) and (27) are verified identically.

V. CONCLUSIONS

In the present paper we have shown that Tomimatsu's integral formulas describe consistently the contribution of magnetic charge in the generalized Smarr mass relation. The physical deficiency of the recent results presented in [8, 9] is therefore explained exclusively by improper use of the expression for the potential A_φ in Tomimatsu's formula defining the angular momentum, which consists in ignoring the fact that A_φ must contain an arbitrary additive constant b_0 whose correct choice is vital for all the physical interpretations. It is interesting that this constant must be chosen distinctly in each of the three dyonic solutions considered in our paper, which may lead us to the following conclusions. First, adding the magnetic charge to a single KN black hole does not really complicate the use of Tomimatsu's formulas, and consequently the extension of Smarr's mass relation, because b_0 must be assigned the same zero value as in the case of the magnetically uncharged KN solution. Second, in the dyonic dihole case the constant b_0 is compelled to take a non-zero value in order to cancel out the term $Q\mathcal{B}$ in Tomimatsu's formula (7) that arises due to the interaction of dyons; it is clear as well that the above term requires the presence of both electric and magnetic charges, so that in the absence of one of these the constant b_0 becomes zero. Third, the dyonic BM solution has revealed that the constant b_0 can also be a multi-valued quantity depending on the number of domains used to determine the potential A_φ, and Tomimatsu's formulas play an important role in finding the particular values of such b_0.

It is remarkable that in all our examples of dyonic spacetimes the generalized mass formula takes a very simple form (27) only slightly different from the usual Smarr's formula. This is a consequence of the relation $\Phi^H/Q = \Phi_m^H/\mathcal{B}$ existing between the potentials Φ^H and Φ_m^H in those particular examples, so that it would be interesting to clarify in the future how generically the above relation holds in the multi-dyonic configurations possessing less symmetry.

Acknowledgments

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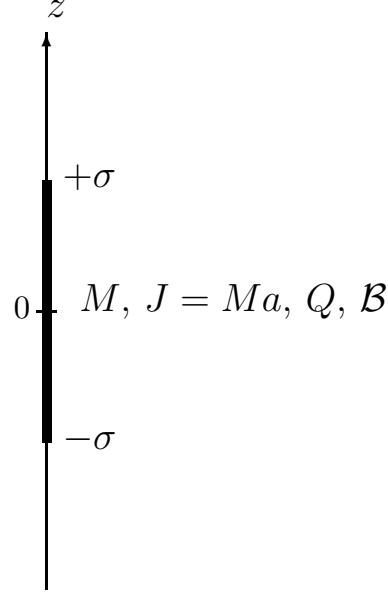[28] I. Semiz, Class. Quantum Grav. **7**, 353 (1990).

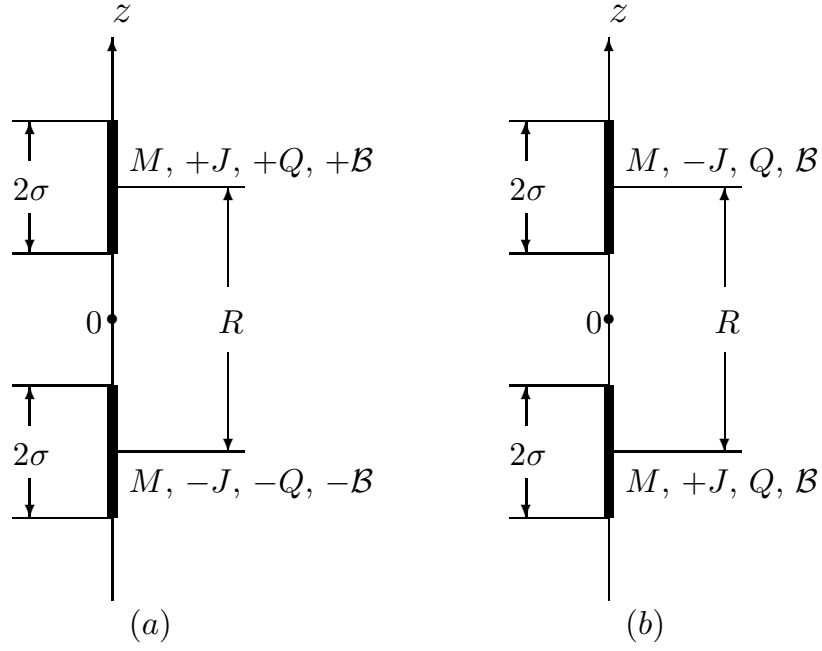FIG. 1: Location of the KN dyon on the symmetry axis.

FIG. 2: Location of the dyonic black holes on the symmetry axis (a) in the case of the EMR dyonic solution, and (b) in the case of the dyonic BM solution.
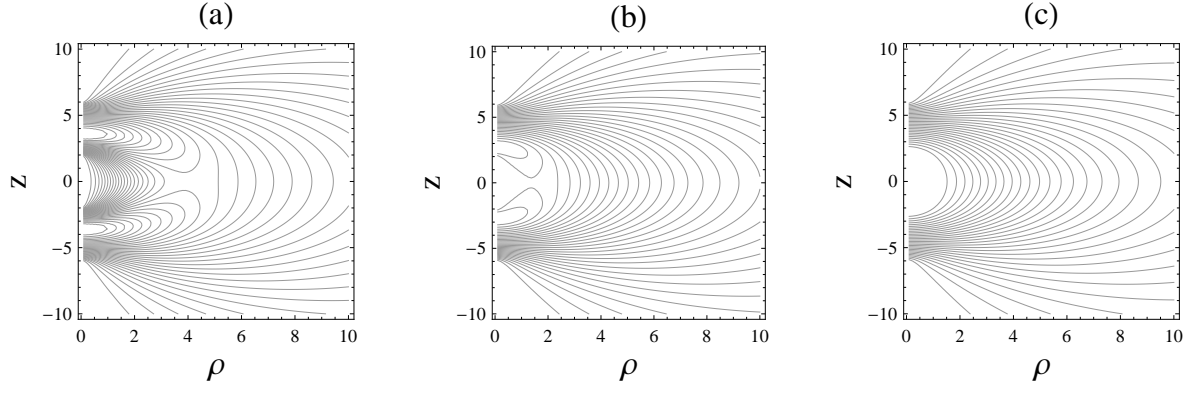
(a) (b) (c)

FIG. 3: Magnetic lines of force in the case of the EMR solution. The particular values of the parameters are the following: $M = 2$, $a = 0.25$, $Q = 0.5$, $R = 8$ (for all plots), and $\mathcal{B} = 0$, 0.02, 0.04 for (a), (b) and (c), respectively.